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EXTREME POINTS OF THE CLASS OF DISCRETE DECREASING FAILURE RATE--ETC(U)

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AFOSR-78-3678

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FSU-STATISTICS-4483

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6 EXTREME POINTS OF THE CLASS OF DISCRETE
DECREASING FAILURE RATE LIFE DISTRIBUTIONS.

by

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9 Interim rept.

FSU Statistics Report M483
AFOSR Technical Report No. 78-4

12 20p.

11 Nov 1978

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

14 FSU-STATISTICS-M483
TR-78-4-AFOSR

¹Research sponsored by the Air Force Office of Scientific Research, AFSC,
USAF, under Grant AFOSR-78-3678

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 79-0094	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Extreme Points of the Class of Discrete Decreasing Failure Rate Life Distributions		5. TYPE OF REPORT & PERIOD COVERED Interim
7. AUTHOR(s) N. A. Langberg R. V. León F. Proschan James Lynch		6. PERFORMING ORG. REPORT NUMBER FSU Statistics Report M483
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306		8. CONTRACT OR GRANT NUMBER(s) AFOSR 78-3678
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research /NM Bolling Air Force Base, D.C. 20332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE November, 1978
		13. NUMBER OF PAGES 20
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Discrete decreasing failure rate, extreme point, mixture of distributions, reliability, nonparametric class of distributions.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We show that the class of discrete decreasing failure rate (DFR) life distributions is a convex class. We then obtain the extreme points of this class. Finally we show how to represent any discrete DFR distribution as a mixture of these extreme points.		

**Extreme Points of the Class of Discrete
Decreasing Failure Rate Life Distributions**

by

Naftali A. Langberg, Ramón V. León, James Lynch, and Frank Proschan

ABSTRACT

We show that the class of discrete decreasing failure rate (DFR) life distributions is a convex class. We then obtain the extreme points of this class. Finally we show how to represent any discrete DFR distribution as a mixture of these extreme points.

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1. Introduction and Summary.

Although a great deal of research has been performed on the class of DFR life distributions in the continuous case (see, e.g., Barlow and Proschan, 1975, Chaps. 3 and 4), very little has been done in the discrete case. Discrete DFR life distributions govern (a) in the grouped data case, the number of periods until failure of a device governed by a DFR life distribution, (b) the number of seasons a TV show is run before being cancelled. Thus discrete DFR life distributions are of great significance in spite of their relative neglect in the reliability literature.

In Section 3 we show that the class of discrete DFR distributions is a convex class and we identify each extreme point of the class. Roughly speaking, the extreme points are the piecewise geometric distributions supplemented by the distribution degenerate at the origin.

In Section 4 we show how to represent by constructive methods any discrete DFR distribution as a mixture of extreme points of the class of discrete DFR distributions.

Note that although the problem treated and the results obtained are of the same type as those of Langberg, León, Lynch, and Proschan (1978) (which obtains the extreme points of the class of DFR distributions in the continuous case and shows that every such DFR distribution is a mixture of the extreme points of this class), the mathematical methods used in the two papers are completely different. Thus in the present paper, the representation of a member of the discrete DFR class as a mixture of

extreme points is obtained by constructive methods; the existence of a representation for the continuous case treated in the referenced paper is obtained by the use of Choquet's Theorem [see Phelps 1966, pp. 19-20].

Having obtained the extreme points of the class of discrete DFR distributions, we plan to use this information to obtain bounds, inequalities, and optimal values of convex functionals of discrete DFR distributions.

2. Preliminaries.

A distribution F is a discrete life distribution if its support is contained in the set $\{0, 1, \dots\}$. Throughout, we denote the survival function of a discrete life distribution F by $\bar{P}_F(k) \equiv 1 - F(k - 1)$, $k = 0, 1, \dots$.

Definition 2.1. Let F be a discrete life distribution. Then F is decreasing failure rate (DFR) if $\bar{P}_F^2(k + 1) \leq \bar{P}_F(k) \bar{P}_F(k + 2)$ for $k = 0, 1, \dots$.

We note that

Proposition 2.2. Let F be a discrete DFR life distribution. Then (a) the support of F is the set $\{0, 1, \dots\}$, or (b) the support of F is the set $\{0\}$.

Let F_d be the discrete life distribution having support $\{0\}$. We denote the failure rate of a discrete DFR life distribution $F \neq F_d$ by

$$r_F(k) \equiv \frac{\bar{P}_F(k) - \bar{P}_F(k + 1)}{\bar{P}_F(k)}, \quad k = 0, 1, \dots, \text{ and denote the class of discrete}$$

DFR life distributions by G_D .

We define two basic concepts to be used in the paper.

Definition 2.3. Let G be a class of distribution functions. Then G is a convex class if $F \equiv \theta F_1 + (1 - \theta)F_2$ belongs to G whenever F_1 and F_2 belong to G and θ is in $[0, 1]$.

Definition 2.4. Let G be a convex class of distribution functions, and let F be in G . Then F is an extreme point of G if there exist no distribution functions F_1 and F_2 in G and a real number θ in $(0, 1)$ such that $F \equiv \theta F_1 + (1 - \theta)F_2$.

Throughout we define a product over an empty set of indices as 1, a sum over an empty set of indices as zero, and a minimum over an empty set as ∞ .

3. The Extreme Points of the DFR Class.

In this section we identify the extreme points of the class of discrete DFR life distributions.

We note that

Proposition 3.1. A discrete DFR life distribution $F \neq F_d$ is DFR iff $r_F(k) \geq r_F(k+1)$ for $k = 0, 1, \dots$.

Proposition 3.2. Let k be a nonnegative integer and let $F \neq F_d$ be a discrete DFR life distribution. Then $r_F(k) = r_F(k+1)$ iff $\bar{p}_F^2(k+1) = \bar{p}_F(k) \bar{p}_F(k+2)$.

First we show that the class G_D is convex.

Lemma 3.3. The class of discrete DFR life distributions is convex.

Proof. Let $F \equiv \theta F_1 + (1 - \theta)F_2$, where F_1 and F_2 belong to G_D and θ is in $[0, 1]$. We show that F is a discrete DFR life distribution. Let k be in the set $\{0, 1, \dots\}$. Then

$$\bar{p}_F(k) \bar{p}_F(k+2) = [\theta \bar{p}_{F_1}(k) + (1 - \theta)\bar{p}_{F_2}(k)] \cdot [\theta \bar{p}_{F_1}(k+2) + (1 - \theta)\bar{p}_{F_2}(k+2)].$$

By algebraic simplification we obtain that

$$\bar{p}_F(k) \bar{p}_F(k+2) \geq [\theta \sqrt{\bar{p}_{F_1}(k) \bar{p}_{F_1}(k+2)} + (1 - \theta) \sqrt{\bar{p}_{F_2}(k) \bar{p}_{F_2}(k+2)}]^2.$$

$$\text{Since } [\theta \sqrt{\bar{p}_{F_1}(k) \bar{p}_{F_1}(k+2)} + (1 - \theta) \sqrt{\bar{p}_{F_2}(k) \bar{p}_{F_2}(k+2)}]^2$$

$$\geq [\theta \bar{p}_{F_1}(k+1) + (1 - \theta)\bar{p}_{F_2}(k+1)]^2 = \bar{p}_F(k+1),$$

the conclusion of the lemma follows. ||

To accomplish the objective of this section we need the following lemma:

Lemma 3.4. Let $F \equiv \theta F_1 + (1 - \theta)F_2$, where $F_1 \neq F_d$ and $F_2 \neq F_d$ belong to G_D , and θ is in $[0, 1]$. Assume that for some nonnegative integer k , $r_F(k) = r_F(k + 1)$. Then $r_F(k + j) = r_{F_1}(k + j) = r_{F_2}(k + j)$ for $j = 0, 1$.

Proof. In the proof of Lemma 3.3 we establish the following chain of inequalities:

$$\begin{aligned} \bar{P}_F(k) \bar{P}_F(k + 2) &\geq [\theta \sqrt{\bar{P}_{F_1}(k) \bar{P}_{F_1}(k + 2)} + (1 - \theta) \sqrt{\bar{P}_{F_2}(k) \bar{P}_{F_2}(k + 2)}]^2 \\ &\geq [\theta \bar{P}_{F_1}(k + 1) + (1 - \theta) \bar{P}_{F_2}(k + 1)]^2 = \bar{P}_F^2(k + 1). \end{aligned}$$

Since by the hypothesis of the present lemma and Proposition 3.2, the two extreme values in the chain are equal, the following equalities hold:

(i) $\bar{P}_{F_j}^2(k + 1) = \bar{P}_{F_j}(k) \bar{P}_{F_j}(k + 2)$ for $j = 1, 2$, and (ii) $\bar{P}_{F_j}(k) = A \bar{P}_{F_j}(k + 2)$ for $j = 1, 2$, and for some number $A \geq 1$. Consequently the conclusion of the lemma follows. ||

Next we identify the extreme points of the class of discrete DFR life distributions. Let $G_{D,e} \equiv \{F_d\} \cup \{F | F \in G_D, 1 > r_F(0) = r_F(1), \text{ and } r_F(k) = r_F(k - 1), \text{ or } r_F(k) = r_F(k + 1), k = 2, 3, \dots\}$. We prove the following:

Theorem 3.5 $G_{D,e}$ is the class of all extreme points in the class of discrete DFR life distributions.

Proof. First we show that all life distributions in $G_{D,e}$ are extreme points. Let $F \equiv \theta F_1 + (1 - \theta)F_2$, where F_1 and F_2 belong to G_D , F belongs to $G_{D,e}$, and θ is in $(0, 1)$. To show that F is an extreme point in G_D , it suffices to prove that $F_1 = F_2$. Clearly F_d is an extreme point. Thus, to prove that F is an extreme point it suffices to consider two cases:

(a) $F \neq F_d$, $F_1 \neq F_d$, and $F_2 \neq F_d$, and (b) $F \neq F_d$, $F_1 \neq F_d$, and $F_2 = F_d$.

(a) Let us assume that $F \neq F_d$, $F_1 \neq F_d$, and that $F_2 \neq F_d$. Since F belongs to $G_{D,e}$, then (i) $\bar{P}_F(1) = \bar{P}_F(2)$ and (ii) $\bar{P}_F^2(k+1) = \bar{P}_F(k)\bar{P}_F(k+2)$ or $\bar{P}_F^2(k+2) = \bar{P}_F(k+1)\bar{P}_F(k+3)$ for $k = 1, 2, \dots$. Hence it follows from Lemma 3.4 that $F_1 = F_2$.

(b) Next let us assume that $F \neq F_d$, $F_1 \neq F_d$, and $F_2 = F_d$. Then $\bar{F}(k) = \theta \bar{F}_1(k)$ for $k = 0, 1, \dots$. Hence, $1 - r_F(0) = \theta[1 - r_{F_1}(0)]$, and $1 - r_F(0) = 1 - r_{F_1}(1)$. The last two equalities imply that $r_{F_1}(0) < r_{F_1}(1)$, a contradiction.

Consequently every discrete DFR life distribution in the set $G_{D,e}$ is an extreme point.

To complete the proof of the theorem, we show that a discrete DFR life distribution that does not belong to $G_{D,e}$ is not an extreme point in G_D . To show that a discrete DFR life distribution is not an extreme point, it suffices to prove that the life distribution can be written as a proper convex combination of two distinct discrete DFR life distributions.

Let F be a discrete DFR life distribution that does not belong to $G_{D,e}$.

Then (a) there exists a positive integer k_0 such that $r_F(k_0 - 1) > r_F(k_0) > r_F(k_0 + 1)$, or (b) $1 > r_F(0) > r_F(1) = r_F(2) = \dots$.

(a) Let us assume that there exists a positive integer k_0 such that $r_F(k_0 - 1) > r_F(k_0) > r_F(k_0 + 1)$. Then the failure rates sequences

$$r_1(k) \equiv \begin{cases} r_F(k) & k \neq k_0 \\ r_F(k_0 - 1) & k = k_0 \end{cases}, \text{ and } r_2(k) = \begin{cases} r_F(k) & k \neq k_0 \\ r_F(k_0 + 1) & k = k_0 \end{cases}$$

determine respectively two distinct discrete DFR life distributions F_1 and F_2 .

Note that $F = \theta F_1 + (1 - \theta)F_2$, where $\theta \equiv \frac{r_F(k_0) - r_F(k_0 + 1)}{r_F(k_0 - 1) - r_F(k_0 + 1)}$. Since θ is in $(0, 1)$ we conclude that F is not an extreme point.

(b) Let us assume that $1 > r_F(0) > r_F(1) = r_F(2) = \dots$. Then the failure rate sequence

$$r_3(k) \equiv \begin{cases} r_F(1) & k = 0 \\ r_F(k) & k > 0 \end{cases}$$

determines a discrete DFR life distribution F_3 . Note that

$F = \frac{1 - r_F(0)}{1 - r_F(1)} F_3 + \frac{r_F(0) - r_F(1)}{1 - r_F(1)} F_d$. Since $[1 - r_F(0)]/[1 - r_F(1)]$ is in $(0, 1)$ we conclude that F is not an extreme point.

Thus, the conclusion of the theorem follows. ||

4. Representation of Discrete DFR Life Distributions as a Mixture of Extreme Points.

In this section we prove that every discrete DFR life distribution can be represented as a mixture of the extreme points of the DFR class. More explicitly, for any discrete DFR life distribution F , we first define a probability measure W_F supported by a subset of the DFR class of extreme points. We then prove that F is a mixture, with weights determined by W_F , of the extreme points that belong to the support of W_F .

Let F be a discrete DFR life distribution, and let F_1 be a discrete DFR life distribution with a failure rate sequence given by

$$r_{F_1}(k) \equiv \begin{cases} r_F(1) & k = 0 \\ r_F(k) & k > 0. \end{cases}$$

Then

$$(4.1) \quad F = \frac{1 - r_F(0)}{1 - r_F(1)} F_1 + \frac{r_F(0) - r_F(1)}{1 - r_F(1)} F_d.$$

Thus, in order to define a corresponding probability measure W_F , it suffices to consider distributions in the set $G_0 \equiv \{F \mid F \in G_D, r_F(0) = r_F(1)\}$, and then use Equation (4.1) to determine W_F for every discrete DFR life distribution F .

Let F be in the set G_0 . To define a probability measure W_F we construct first a vector of random variables $\underline{Y}(F, \cdot) \equiv (Y_1(F, \cdot), Y_2(F, \cdot), \dots)$, with components assuming values in the set $\{0, 2, 3, \dots, \infty\}$, defined on some space Ω . Then we define for each $w \in \Omega$ a discrete DFR life distribution $G_{\underline{Y}(F, w)}$ which belongs to the class of the discrete DFR extreme points and depends on the parameter $\underline{Y}(F, w)$. Finally, we prove that

$F(k) = E G_{\underline{Y}(F, \cdot)}(k)$ for $k = 0, 1, \dots$. Thus, the vector $\underline{Y}(F, \cdot)$ determines a probability measure W_F supported by a subset of $G_F \equiv \{G_{\underline{Y}(F, \omega)}, \omega \in \Omega\}$, such that $W_F\{G_{\underline{Y}(F, \omega)} | Y_q(F, \omega) = k_q, q = 1, \dots, s\} \equiv P\{Y_q(F, \cdot) = k_q, q = 1, \dots, s\}$ for k_1, \dots, k_s in the set $\{0, 2, 3, \dots, \infty\}$, and $s = 1, 2, \dots$, and F is a mixture, with weights determined by W_F , of the discrete DFR extreme points which belong to the set G_F .

For illustrative purposes we define the probability measure W_F for two special cases: (a) F is an extreme point, and (b) F has only four distinct failure rates that appear successively.

(a) Let F be an extreme point, let ℓ be the number (possibly infinite) of distinct failure rates of F , and let $k_q, 1 \leq q < \min(\ell + 1, \infty)$, be the number of times (possibly infinite) the q th distinct failure rate of F appears as a failure rate of F . We define $Y_q(F, \cdot) \equiv k_q, 1 \leq q < \min(\ell + 1, \infty)$, and $Y_q(F, \cdot) \equiv 0$ for $\infty > q \geq \ell + 1$, provided that $\ell < \infty$. For $\omega \in \Omega$ let $Y_q(F, \omega), 1 \leq q < \min(\ell + 1, \infty)$, denote the number of times $r_F[\sum_{j=1}^{q-1} Y_j(F, \omega) + 1]$ appears as a failure rate in $G_{\underline{Y}(F, \omega)}$. Clearly $F \equiv G_{\underline{Y}(F, \cdot)}$; thus the degenerate probability measure determined by $\underline{Y}(F, \cdot)$ and supported by $\{F\}$ is the desired one.

(b) Let $1 > \alpha(0) > \alpha(2) > \alpha(3) > \alpha(4) > 0$, and let

$$r(k) \equiv \begin{cases} \alpha(0) & k = 0, 1 \\ \alpha(k) & k = 2, 3 \\ \alpha(4) & k \geq 4, \end{cases} \quad \text{be the failure rate sequence of}$$

a discrete DFR life distribution F . Let

$$r_1(k) \equiv \begin{cases} \alpha(0), & k = 0, 1 \\ \alpha(3) & k = 2, 3, \\ \alpha(4) & k \geq 4, \end{cases} \quad r_2(k) \equiv \begin{cases} \alpha(0) & k = 0, 1, 2 \\ & , \text{ and } r_3(k) \equiv \begin{cases} \alpha(0) & k \leq 4 \\ \alpha(4) & k > 4 \end{cases} \\ \alpha(k) & k \geq 3 \end{cases}$$

be the failure rate sequences of the discrete DFR life distributions H_1 , H_2 , and H_3 respectively. Then by repeating twice the construction presented in the proof of Theorem 3.5, we obtain that

$$(4.2) \quad F = \frac{\alpha(0) - \alpha(2)}{\alpha(0) - \alpha(3)} H_1 + \frac{\alpha(2) - \alpha(3)}{\alpha(0) - \alpha(4)} H_2 + \frac{\alpha(3) - \alpha(4)}{\alpha(0) - \alpha(4)} \cdot \frac{\alpha(2) - \alpha(3)}{\alpha(0) - \alpha(3)} H_3.$$

We construct the sequence $\underline{Y}(F, \cdot)$ as follows

$$P\{Y_1(F, \cdot) = j\} \equiv \begin{cases} 2 & \frac{\alpha(0) - \alpha(2)}{\alpha(0) - \alpha(3)} \\ 3 & \frac{\alpha(2) - \alpha(3)}{\alpha(0) - \alpha(4)} \\ 4 & \frac{\alpha(2) - \alpha(3)}{\alpha(0) - \alpha(3)} \cdot \frac{\alpha(3) - \alpha(4)}{\alpha(0) - \alpha(4)}, \end{cases}$$

$$P\{Y_2(F, \cdot) = j | Y_1(F, \cdot) = j\} \equiv \begin{cases} 1 & Y_1(F, \omega) = 2, j = 2 \\ 1 & Y_1(F, \omega) = 3, j = \infty \\ 1 & Y_1(F, \omega) = 4, j = \infty, \end{cases}$$

$Y_{2+q}(F, \cdot) \equiv 0$, $q = 1, 2, \dots$. For $\omega \in \Omega$, $Y_q(F, \omega)$, $q = 1, 2$, denotes the number of times $r_F[\sum_{j=1}^{q-1} Y_j(F, \omega) + 1]$ appears as a failure rate in $G_{\underline{Y}(F, \omega)}$.

Clearly $G_{(2, 2, \infty, 0, 0, \dots)} = H_1$, $G_{(3, \infty, 0, 0, \dots)} = H_2$, and

$G_{(4, \infty, 0, 0, \dots)} = H_3$. From Equation (4.2) it follows that $F(k) = E G_{\underline{Y}(F, \omega)}^{(k)}$ for $k = 0, 1, \dots$. Thus the probability measure determined by $\underline{Y}(F, \cdot)$

and supported by the set of extreme points $\{H_1, H_2, H_3\}$ is the desired one.

Before constructing the vector $\underline{Y}(F, \cdot)$, we define two quantities and prove a lemma needed for the construction. Let F be a discrete DFR life distribution such that $1 > r_F(0) = r_F(1) > r_F(2) > r_F(3)$. We denote by $m(F)$ the $\min\{k | k = 2, 3, \dots, r_F(k) = r_F(k+1)\}$, and by $Z(F, \cdot)$ a random variable which assumes values in the set $\{2, 3, \dots, m(F)\}$ with respective

probabilities equal to $\frac{r_F(0) - r_F(q)}{r_F(0) - r_F(q+1)} \prod_{j=2}^{q-1} \frac{r_F(j) - r_F(j+1)}{r_F(0) - r_F(j+1)}$ for

$2 \leq q < \min[m(F) + 1, \infty]$. Next we justify our claim that $Z(F, \cdot)$ is a random variable.

Lemma 4.1. Let F be a discrete DFR life distribution such that $r_F(0) = r_F(1) > r_F(2) > r_F(3)$. Then (a) $m(F) < \infty$ implies that $\sum_{q=2}^{m(F)} P\{Z(F, \omega) = q\} = 1$, and (b) $m(F) = \infty$ implies that $\sum_{q=2}^{\infty} P\{Z(F, \omega) = q\} \leq 1$.

Proof. Let $3 < m(F) < \infty$, and let

$$r_1(k) \equiv \begin{cases} r_F(k) & k \leq m(F) - 2 \\ r_F(k+1) & k \geq m(F) - 1, \end{cases}$$

be the failure rate sequence of a discrete DFR life distribution F_1 . Then $m(F_1) = m(F) - 1$, and $\sum_{q=2}^{m(F_1)} P\{Z(F_1, \omega) = q\} = \sum_{q=2}^{m(F)} P\{Z(F, \omega) = q\}$. Thus, part (a) of the lemma follows by induction on $m(F)$. Let $m(F) = \infty$, let j be a positive integer in the set $\{3, 4, \dots\}$, and let

$$r_2(k) \equiv \begin{cases} r_F(k) & k \leq j \\ r_F(j) & k = j + 1 \\ r_F(k - 1) & k > j + 1 \end{cases}$$

be the failure rate sequence of a discrete DFR life distribution F_2 . Then $j = m(F_2)$, and $\sum_{q=2}^j P\{Z(F, \omega) = q\} \leq \sum_{q=2}^{m(F_2)} P\{Z(F_2, \omega) = q\}$. Thus, from part (a) of the lemma we obtain that $\sum_{q=2}^j P\{Z(F, \omega) = q\} \leq 1$ for $j = 3, 4, \dots$. Consequently part (b) of the lemma follows. ||

We start to construct $\underline{Y}(F, \cdot)$ by deriving $Y_1(F, \cdot)$. Let F be a discrete DFR life distribution which does not belong to the class of the DFR extreme points such that $r_F(0) = r_F(1)$. Let $j_1 \equiv \min\{k | k = 2, 3, \dots, r_F(k-1) > r_F(k) > r_F(k+1)\}$, and let $r_1(k) \equiv r_F(j_1 + k - 2)$, $k = 0, 1, \dots$, be the failure rate sequence of a discrete DFR life distribution F_1 . We define $Y_1(F, \cdot)$ to have the same distribution as $Z(F_1, \cdot)$.

We proceed to define $Y_2(F, \cdot)$, $Y_3(F, \cdot)$, \dots . If $Y_1(F, \omega) = \infty$, then we define $Y_{q+1}(F, \omega) \equiv 0$, for $q = 1, 2, \dots$. For $Y_1(F, \omega) < \infty$, let $r_1[Y_1(F, \omega) + 1]$, $r_1[Y_1(F, \omega) + 1]$, $r_1[Y_1(F, \omega) + 2]$, \dots be respectively the failure rates of a discrete DFR life distribution $F_{Y_1(F, \cdot)}$. Then $Y_2(F, \cdot) | Y_1(F, \omega)$, and $r_2(k, \omega)$, $k = 0, 1, \dots$, are defined in the same way that we defined $Y_1(F, \cdot)$, and $r_1(k)$, $k = 0, 1, \dots$, where F is replaced by $F_{Y_1(F, \omega)}$. Assume that the joint distribution of $(Y_1(F, \cdot), \dots, Y_d(F, \cdot))$, $d = 2, 3, \dots$, is well defined, and that $Y_q(F, \cdot) | Y_1(F, \omega), \dots, Y_{q-1}(F, \omega)$, $q = 2, \dots, d$, and $Y_q(F, \cdot) | Y_{q-1}(F, \omega)$ have the same distribution. Then we define $Y_{d+1}(F, \cdot)$ in the same way that we defined $Y_2(F, \cdot)$, where $Y_1(F, \omega)$ and r_1 are replaced by $Y_d(F, \omega)$ and r_d respectively.

Next we define for each $\omega \in \Omega$ a discrete DFR life distribution $G_{Y(F, \omega)}$ which belongs to $G_{D,e}$. Let F be a discrete life distribution which

does not belong to the class of the DFR extreme points such that $r_F(0) = r_F(1)$.

Let $j_1 \equiv \min\{k | k = 2, 3, \dots, r_F(k-1) > r_F(k) > r_F(k+1)\}$. Then for every

$\omega \in \Omega$, $r_F(0), \dots, r_F(j_1 - 1)$, are the first j_1 failure rates of $G_{Y(F, \omega)}$.

For $\omega \in \Omega$, $Y_1(F, \omega)$ represents the number of times the failure rate

$r_F(j_2 - 2)$, which is equal to $r_F(j_1 - 1)$, appears in $G_{Y_1(F, \omega)}$. If

$Y_1(F, \omega) = \infty$, then the construction of $G_{Y(F, \omega)}$ is completed. Next let

$Y_1(F, \omega) < \infty$; to determine the failure rates of $G_{Y(F, \omega)}$ for $k = j_1 + Y_1(F, \omega) - 1$,

$j_1 + Y_1(F, \omega), \dots$, we repeat the construction, but replace F by $F_{Y_1(F, \omega)}$.

We continue to construct the discrete DFR life distribution $G_{Y(F, \omega)}$

recursively; in the $(k+1)^{\text{th}}$ step, we repeat the original construction,

but replace F by $F_{Y_k(F, \omega)}$.

Let F be a discrete DFR life distribution if \mathcal{G}_0 . To prove that

$F(k) = E G_{Y(F, \cdot)}(k)$ for $k = 0, 1, \dots$, we need the following:

Lemma 4.2. Let F be a discrete DFR life distribution such that

$r_F(0) = r_F(1) > r_F(2) > r_F(3)$, let X be a random variable, assuming

values in the set $\{2, 3, \dots\}$, and let $\psi(F, X, k) \equiv r_F(k+1)P\{X \geq k+1\} -$

$r_F(k+2)P\{X = k+1\} - r_F(0)P\{X \geq k+2\}$ for $k = 0, 1, \dots$. Then for

$k = 0, 1, \dots$, $\psi[F, Z(F, \cdot), k] = 0$.

Proof. Let $3 < m(F)$, and let $r_1(k) \equiv \begin{cases} r_F(k) & k = 0, 1 \\ r_F(k+1) & k > 1 \end{cases}$,

be the failure rate sequence of a discrete DFR life distribution F_1 . Then

(a) $m(F_1) + 1 = m(F)$, (b) $r_{F_1}(k) = r_F(k+1)$ for $k = 2, 3, \dots$, and

(c) $P\{Z(F_1, \omega) = k\} = \frac{r_F(0) - r_F(3)}{r_F(2) - r_F(3)} \cdot P\{Z(F, \omega) = k+1\}$ for $k = 2, 3, \dots$.

Thus $\psi[F, Z(F, \cdot), k+1] = \frac{r_F(0) - r_F(3)}{r_F(2) - r_F(3)} \psi[F_1, Z(F_1, \cdot), k]$ for $k = 0, 1, \dots$.

The conclusion of the lemma follows by induction on k . ||

We prove now that $F(k) = E G_{Y(F, \omega)}^{(k)}$ for $k = 0, 1, \dots$ and F in G_0 .

Theorem 4.3. Let F be a discrete DFR life distribution that does not belong to the class of the discrete DFR extreme points such that $r_F(0) = r_F(1)$. Then for every nonnegative integer k ,

$$(4.3) \quad \bar{F}(k) = E \bar{G}_{Y(F, \cdot)}^{(k)}$$

Proof. Without loss of generality, assume that $r_F(1) > r_F(2) > r_F(3)$. Clearly Equation (4.3) holds for $k = 0, 1$, and 2. Assume that Equation (4.3) holds for all nonnegative integers which are smaller than a fixed $k + 1$. We prove that Equation (4.3) holds for $k + 1$.

Denote by $r_G(\omega, q)$, $q = 0, 1, \dots$, the q th failure rate of $G_{Y(F, \cdot)}$.

By our assumption and by construction of $G_{Y(F, \cdot)}$, we obtain that

$$I[Y_1(F, \omega) \leq k] \cdot E \left\{ \prod_{q=Y_1(F, \omega)}^k [1 - r_G(\omega, q)] \mid Y_1(F, \omega) \right\} =$$

$$I[Y_1(F, \omega) \leq k] [1 - r_F\{Y_1(F, \omega) + 1\}] \cdot \prod_{q=1}^{k-Y_1(F, \omega)} [1 - r_F(q + Y_1(F, \omega))].$$

Thus, for $k = 2, 3, \dots$,

$$(4.4) \quad E \bar{G}_{Y(F, \cdot)}^{(k)} = [1 - r_F(0)]^{k+1} P(Y_1(F, \omega) \geq k + 1) +$$

$$\sum_{q=2}^k P(Y_1(F, \omega) = q) [1 - r_F(0)]^q [1 - r_F(q + 1)] \prod_{j=1}^{k-q} [1 - r_F(q + j)].$$

By our assumption, and Equation (4.4), we obtain that

$$E \bar{G}_{Y(k, \cdot)}^{(k+1)} = \bar{F}(k+1) + [1 - r_F(0)]^{k+1} \phi[F, Y_1(F, \cdot), k].$$

The conclusion of the theorem follows from Lemma 4.2. ||

Next we state and prove the main result of this section.

Theorem 4.4. Let F be a discrete DFR life distribution. Then we can construct a probability measure W_F , supported by a subset of the discrete DFR class of extreme points, such that F is a mixture, with weights determined by W_F , of the DFR extreme points in the support of W_F .

Proof. Clearly the result holds for every discrete DFR life distribution that belongs to $G_{D,e}$. Let F be a discrete DFR life distribution which does not belong to $G_{D,e}$, and let F_1 be as in Equation (4.1). Then we

define W_F as follows: $W_F\{F_d\} = \frac{r_F(0) - r_F(1)}{1 - r_F(1)}$, and $W_F\{G_{Y(F_1, \omega)} | Y_q(F_1, \omega) = k_q, q = 1, \dots, s\} = \frac{1 - r_F(0)}{1 - r_F(1)} W_{F_1}\{G_{Y(F_1, \omega)} | Y_q(F_1, \omega) = k_q, q = 1, \dots, s\}$

for k_1, \dots, k_s in the set $\{2, 3, \dots, \infty\}$, and $s = 1, 2, \dots$. From Theorem 4.3 and Equation (4.1) it follows that W_F is a probability measure with a support in the subset $\{F_d\} \cup \{G_{Y(F_1, \omega)} | \omega \in \Omega\}$ of DFR extreme points such that F is a mixture, with weights determined by W_F , of the extreme points in the support of W_F . ||

Finally we note that, as is frequently the case, the representation is not unique. To show that the representation is not unique, let

$$r(k) = \begin{cases} \alpha(0) & k = 0 \\ \alpha(1) & k \geq 1, \end{cases}$$

$r_1(k) = \alpha(1), k = 0, 1, \dots$, and let

$$r_2(k) = \begin{cases} \frac{\alpha(0) + 1}{2} & k = 0 \\ \alpha(1) & k \geq 1, \end{cases}$$

be the failure rate sequences of the discrete

DFR life distributions F , F_1 , and F_2 respectively, where $1 > \alpha(0) > \alpha(1) > 0$.

Then

$$(4.5) \quad F = \frac{1 - \alpha(0)}{\alpha(0) + 1 - 2\alpha(1)} F_1 + \frac{2[\alpha(0) - \alpha(1)]}{\alpha(0) + 1 - 2\alpha(1)} F_2.$$

Thus F has two representations, one based on Equation (4.1), and the other on Equation 4.5.

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